

CONVEX DECOMPOSITION THEORY

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ABSTRACT. We use convex decomposition theory to (1) reprove the existence of a universally tight contact structure on every irreducible 3-manifold with nonempty boundary, and (2) prove that every toroidal 3-manifold carries infinitely many nonisotopic, nonisomorphic tight contact structures.

It has been known for some time that there are deep connections between the theory of taut foliations and tight contact structures due to the work of Eliashberg and Thurston [7]. In particular, they proved that a taut foliation can be perturbed into a (universally) tight contact structure. In a previous paper [19] we introduced the notion of *convex decompositions* and explained how convex decompositions can naturally be viewed as generalizations of *sutured manifold decompositions* introduced by Gabai in [8]. In this paper we take the viewpoint that convex decompositions are completely natural in 3-dimensional contact topology and that many theorems can be proven directly in the category of tight contact manifolds with convex splittings as morphisms.

The first theorem of the paper is a version of a theorem by Gabai-Eliashberg-Thurston in the case of manifold with boundary:

Theorem 0.1 (Gabai-Eliashberg-Thurston). *Let (M, γ) be an oriented, compact, connected, irreducible, sutured 3-manifold which has boundary, is taut, and has annular sutures. Then (M, γ) carries a universally tight contact structure.*

We provide an alternate proof which (1) does not require us to perturb taut foliations into tight contact structures as we did in [19] and (2) does not resort to four-dimensional *symplectic filling* techniques in order to prove tightness. Instead, we use Gabai's sutured manifold decomposition and directly apply a gluing theorem (Theorem 1.6). To prove the gluing theorem we apply key ideas from Colin's papers [3] and [4] in the context of convex decomposition theory.

We also apply similar ideas to prove the following theorem:

Theorem 0.2. *Let M be an oriented, closed, connected, irreducible 3-manifold which contains an incompressible torus. Then M carries infinitely many isomorphism classes of universally tight contact structures.*

This theorem confirms a conjecture which has its beginnings in the works of Giroux [10] and Kanda [23], was proved for torus bundles over S^1 by Giroux [11], and was extended

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by Colin [4] to the case where there exist two incompressible tori T, T' with a “persistent” intersection. The flip side of Theorem 0.2 would be the following conjecture:

Conjecture 0.3. *Let M be a closed, connected, irreducible 3-manifold which is atoroidal (does not contain an incompressible torus). Then M carries only finitely many isotopy classes of tight contact structures.*

It should be noted that this work has been influenced greatly by the work of Colin [3, 4, 5]. In particular, the reader familiar with Colin’s work will recognize that many of the ideas of this paper are adaptations and strengthenings of Colin’s ideas in the setting of convex decompositions. We have also been informed by Colin that he has recently obtained Theorem 0.2 (independently).

Necessary background. This paper is intended as a sequel to [19], and it is recommended that the reader read it first. We will freely use the circle of ideas introduced there. The reader will also find it helpful to have read [16] and [18] and to have familiarized him/herself with ideas of gluing and bypasses.

Conventions.

1. M = oriented, compact 3-manifold.
2. ξ = positive contact structure which is co-oriented by a global 1-form α .
3. A convex surface Σ is either closed or compact with Legendrian boundary.
4. Γ_Σ = dividing set of a convex surface Σ .
5. $\#\Gamma_\Sigma$ = number of connected components of Γ_Σ .
6. $|\beta \cap \gamma|$ = geometric intersection number of two curves β and γ on a surface.
7. $\#(\beta \cap \gamma)$ = cardinality of the intersection.
8. $\Sigma \setminus \gamma$ = metric closure of the complement of γ in Σ .
9. $t(\beta, Fr_S)$ = twisting number of a Legendrian curve with respect to the framing induced from the surface S .

1. GLUING AND PROOF OF THEOREM 0.1

In this paper we will assume that the reader is familiar with concepts of sutured manifold decompositions and convex decompositions. In particular, we refer the reader to [19] for an explanation of the relationship between sutured manifold decompositions and convex decompositions. We will also need the following definitions.

Definition 1.1. *A sutured manifold with annular sutures is a sutured manifold (M, γ) which satisfies the following:*

1. ∂M is nonempty.
2. Every component of ∂M contains an annular suture.
3. Every component of γ is an annulus, that is $\gamma = A(\gamma) \neq \emptyset$.

Definition 1.2. *Let S be a properly embedded compact convex surface with Legendrian boundary. A connected component γ of Γ_S is boundary-parallel (or ∂ -parallel) if it is an arc with boundary on ∂S that cuts off a half-disk D from S . We also say that Γ_S is ∂ -parallel if all of its connected components are ∂ -parallel.*

Let (M_0, γ_0) be a sutured manifold with annular sutures and let

$$(M_0, \gamma_0) \xrightarrow{S_0} (M_1, \gamma_1) \xrightarrow{S_1} \cdots \xrightarrow{S_{n-1}} (M_n, \gamma_n) = \cup(B^3, S^1 \times I),$$

be a sutured manifold hierarchy. We define the corresponding convex hierarchy

$$(M_0, \Gamma_0) \xrightarrow{(S_0, \sigma_0)} (M_1, \Gamma_1) \xrightarrow{(S_1, \sigma_1)} \cdots \xrightarrow{(S_{n-1}, \sigma_{n-1})} (M_n, \Gamma_n) = \cup(B^3, S^1),$$

as follows. Let (M_i, Γ_i) be the convex structure associated to (M_i, γ_i) , that is Γ_i consists of the oriented cores of γ_i . We may assume that each component of S_i which is contained in γ_i intersects Γ_i transversely in at least two points. Each σ_i is defined to be a collection of ∂ -parallel arcs in S_i such that the half-disks they bound in S_i intersect Γ_i in one point each and have the opposite orientation as S_i , and the orientation induced by the half-disks on the components of σ_i cause σ_i to start in $R_-(\Gamma_i)$ and end in $R_+(\Gamma_i)$. With these conventions it is straightforward to see that (M_i, Γ_i) split along (S_i, σ_i) is (M_{i+1}, Γ_{i+1}) .

For the purposes of a later argument we need to know that we can assume that each component of each ∂S_i intersects the dividing curves Γ_i at least twice. To show how this may be arranged, consider the important special case where some component R of $\partial M_i \setminus \Gamma_i$ contains a component C of ∂S_i such that C does not separate R , and R intersects no other components of ∂S_i . S_i may be perturbed in a neighborhood of C by a “finger move” in two different ways, S'_i, S''_i (see Figure 1). As described above, add appropriate ∂ -parallel dividing curves to S'_i and S''_i . Splitting along precisely one of (S'_i, σ'_i) and (S''_i, σ''_i) will produce the convex structure (M_{i+1}, Γ_{i+1}) , and we continue to abuse notation by denoting this splitting surface (S_i, σ_i) . This technique may also be applied if C is replaced by a family of parallel coherently oriented curves in R . In general, by Theorem 1.3 stated below, we may always assume this is the case by choosing the original sequence S_1, S_2, \dots to be “well-groomed”.

Theorem 1.3 (Gabai). *Let (M, γ) be a taut sutured manifold with $H_2(M, \partial M) \neq 0$. Then (M, γ) admits a sutured manifold hierarchy such that $S_i \cap \partial M_i \neq \emptyset$ if $\partial M_i \neq \emptyset$, and S_i is well-groomed along ∂M_i , i.e., for every component $R \subset \partial M_i \setminus \gamma_i$, $S_i \cap R$ is a union of parallel oriented nonseparating simple closed curves if R is nonplanar and arcs if R is planar.*

We then have the following theorem:

Theorem 1.4. *Let (M, γ) be a sutured manifold with annular sutures. If (M, γ) admits a sutured manifold hierarchy, then there exists a universally tight contact structure on M with convex boundary and dividing set Γ corresponding to γ .*

Proof. Note that existence of a sutured manifold hierarchy implies that (M, γ) is taut. Use Theorem 1.3 to choose a well-groomed sutured manifold hierarchy for (M, γ) and consider the associated convex hierarchy of (M, Γ) . Starting with the last term of the convex hierarchy, $\cup(B^3, S^1)$, inductively define contact structures on the terms of the convex hierarchy. For the initial step, we use the following fundamental result of Eliashberg [6]:

Theorem 1.5 (Eliashberg). *Assume there exists a contact structure ξ on a neighborhood of ∂B^3 which makes ∂B^3 convex with $\#\Gamma_{\partial B^3} = 1$. Then there exists a unique extension of ξ to a tight contact structure on B^3 up to an isotopy which fixes the boundary.*

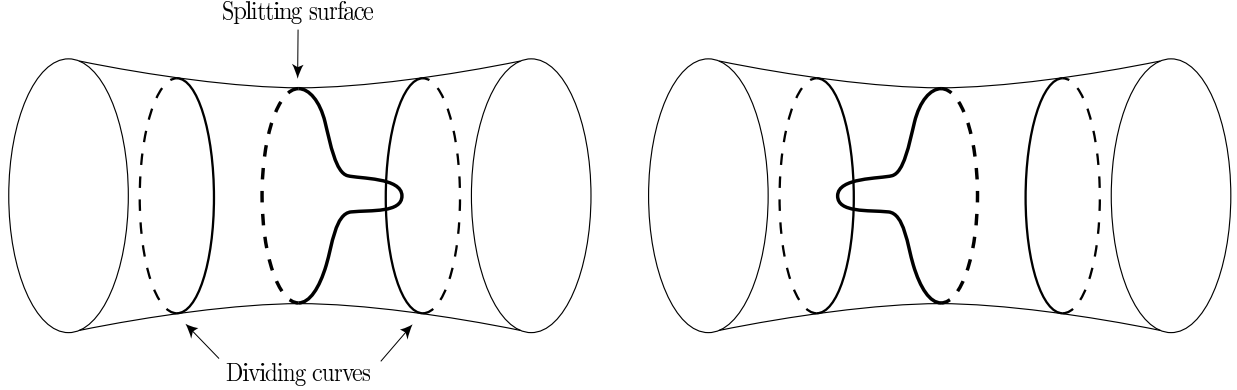


FIGURE 1.

Now that we have unique tight contact structures on the 3-balls, we use the Gluing Theorem (Theorem 1.6) below to successively obtain a universally tight contact structure on the glued-up manifold. The hypotheses that (M, γ) is a sutured manifold with annular sutures and the choice of perturbation of the well-groomed splitting surfaces used in the convex hierarchy of (M, Γ) guarantee that the hypotheses of Theorem 1.6 are satisfied. \square

Note that a splitting surface of a convex splitting associated to a sutured manifold splitting has a ∂ -parallel dividing set. This is crucial as we are about to use the following Gluing Theorem to finish our proof. This theorem first appeared in Colin [3] in a slightly different form. Our formulation is in terms of convex decompositions, and our proof relies on bypasses and dividing curve configurations.

Theorem 1.6 (Gluing). *Let (M, ξ) be an irreducible contact manifold with nonempty convex boundary, and let $S \subset M$ be a properly embedded compact convex surface with Legendrian boundary such that (1) S is incompressible in M , (2) $t(\gamma, Fr_S) < 0$ for each connected component γ of ∂S , i.e., each component of γ nontrivially intersects the dividing set $\Gamma_{\partial M}$, and (3) Γ_S is ∂ -parallel. Consider a decomposition of (M, ξ) along S . If ξ is universally tight on $M \setminus S$, then ξ is universally tight on M .*

Proof. We will prove that, under conditions (1)–(3), the fact that ξ is universally tight on $M \setminus S$ implies that ξ is tight on M . The proof, as will be clear, easily adapts to any finite cover of M , hence showing that the universal cover is tight, provided M is *residually finite* (which is the case, since M is Haken — see Section 4.1.1 for a discussion of residual finiteness). Assume, on the contrary, that M is not tight and that D is an overtwisted disk in M . Then, after a possible contact isotopy, we can assume that D intersects S transversally along Legendrian curves and arcs and that $\partial D \cap S \subset \Gamma_S$ (see Lemma 2.7 in [18]). Note that closed curves in $D \cap S$ are homotopically trivial on S since S is incompressible. We would like to argue that, starting with innermost closed curves, we can eliminate them by pushing S across D (we do not push D across S to avoid introducing self-intersections into D). Since M is assumed to be irreducible, the 2-sphere formed by two disks (one on D and one on S) bounding an innermost curve of intersection δ on S bounds a ball across which S can be isotoped. We will include the sketch of the following fact (see Lemma 2.8 and 2.9 in [18]), as we will need to refer to some steps in the proof of it later.

Lemma 1.7. *We can push S across D to eliminate δ in a finite number of steps, each of which is a bypass along an arc of the circle δ .*

Proof. Consider the subdisk D_δ of D bounded by δ . Since δ is a homotopically trivial Legendrian curve on S , $t(\gamma, Fr_S)$ must be negative. After possible perturbation rel boundary, D_δ is convex with Legendrian boundary satisfying $t(\partial D_\delta) < 0$. Γ_{D_δ} consists of properly embedded arcs with endpoints on ∂D_δ and no closed components, since the interior of D_δ lies in the tight manifold $M \setminus S$. We push S to engulf a bypass along δ corresponding to a ∂ -parallel dividing curve on D_δ . We can continue until there is only one arc left. It is not hard then to see that the last bit of the isotopy is a contact isotopy. \square

In a similar manner we can eliminate outermost arcs of intersection, by engulfing the half-disk of D bounded by such an arc in a ball, and then pushing S across that ball as in the previous lemma. Therefore, starting with innermost circles δ and outermost arcs, we can push D across S to reduce $\#(S \cap D)$ in steps, at the expense of modifying the dividing curve configuration on S . Changes in dividing curve configurations can be reduced to a sequence of bypass attachment moves which must now be analyzed.

Without the assumption that the dividing set on the cutting surface consists of ∂ -parallel arcs the following lemma would not be true.

Lemma 1.8. *Let S be a convex surface with Legendrian boundary in a contact manifold (M, ξ) , such that Γ_S is ∂ -parallel and $(M \setminus S, \xi)$ is tight. Then any convex surface S' obtained from S by a sequence of bypasses will have $\Gamma_{S'}$ obtained from Γ_S by possibly adding pairs of parallel nontrivial curves (up to isotopy rel boundary).*

Proof. This follows by examining the possible bypasses. One possibility is that the bypass is trivial, that is it is contained in a disk and produces no change in the dividing curves. Some bypasses will produce an overtwisted disk, and hence cannot exist inside a tight manifold. Some bypasses introduce a pair of parallel curves, and finally a bypass may change one pair of parallel curves into another pair or remove a pair of parallel curves. \square

We need one more simple lemma:

Lemma 1.9. *Let S be a convex surface with Legendrian boundary in a contact manifold (M, ξ) , such that Γ_S is ∂ -parallel. If a convex surface S' is obtained from S by a bypass such that $\Gamma_{S'}$ is isotopic to Γ_S , then S and S' are contact isotopic, and in particular $(M \setminus S, \xi)$ is tight if and only if $(M \setminus S', \xi)$ is.*

Proof. Left to the reader. \square

Consider a single bypass move on S with Γ_S ∂ -parallel. It is either trivial or increases $\#\Gamma$. If $\#\Gamma$ is increased, the arc of attachment δ starts on an arc l of Γ_S and comes back to l , thereby generating a non-trivial element of $\pi_1(S, l)$. There exists a large enough finite cover $\pi : \widetilde{M} \rightarrow M$ which “expands” S to $\widetilde{S} = \pi^{-1}S$, such that a lift of δ becomes a trivial arc of attachment, that is, it connects two different components of $\Gamma_{\widetilde{S}}$. The existence of the large finite cover follows from the facts that (1) an incompressible surface S has $\pi_1(S)$ which injects into $\pi_1(M)$ and (2) a Haken 3-manifold has residually finite π_1 .

Therefore, by passing through a finite succession of covers, we construct a cover \widetilde{M} , together with the preimage \widetilde{S} of S and a lift \widetilde{D} of the overtwisted disk D , in which all the bypasses needed to isotop \widetilde{S} across \widetilde{D} are trivial. The surface \widetilde{S}' with $\widetilde{S}' \cap \widetilde{D} = \emptyset$, obtained from \widetilde{S} via trivial bypass attachments, has the same dividing set as \widetilde{S} and is contact isotopic to \widetilde{S} by Lemma 1.9. It follows that $(\widetilde{M} \setminus \widetilde{S}', \pi^* \xi)$ is tight. This contradicts the existence of an overtwisted disk $\widetilde{D} \hookrightarrow \widetilde{M} \setminus \widetilde{S}'$ and finishes the proof of Theorem 1.6. \square

Combining Theorem 1.3 with Theorem 1.4, we have Theorem 0.1.

2. TIGHT CONTACT STRUCTURES ON $T^2 \times I$ AND FOLDING

2.1. Universally tight contact structures on $T^2 \times I$. In this section we review key properties of universally tight contact structures on $T^2 \times I$. Details of assertions can be found in [16] (also see [12]).

Fix an oriented identification $T^2 \xrightarrow{\sim} \mathbb{R}^2/\mathbb{Z}^2$. Consider $T^2 \times I = T^2 \times [0, 1]$ with coordinates $((x, y), z)$. We will focus on tight contact structures on $T^2 \times I$ with convex boundary which satisfy

$$(1) \quad \text{slope}(\Gamma_{T_0}) = \text{slope}(\Gamma_{T_1}).$$

Here we write $T_z = T^2 \times \{z\}$. The following proposition is a consequence of the classification of tight contact structures on $T^2 \times I$.

Proposition 2.1. *All tight contact structures satisfying Equation (1) are universally tight.*

The prototypical universally tight contact structures on $T^2 \times I$ are ξ_k , $k \in \mathbb{Z}^+$, given by 1-forms $\alpha_k = \sin(\pi kz)dx + \cos(\pi kz)dy$, with the boundary adjusted so it becomes convex with $\#\Gamma_{T_i} = 2$, $i = 0, 1$. If we pick a curve that meets each component of Γ_{T_0} in a point and take the l -fold cover corresponding to this curve, we call the pull-back contact structure ξ_k as well. Note that for this version of ξ_k we have $\#\Gamma_{T_i} = 2l$.

We say that a curve C on a convex surface Σ is *efficient* (with respect to Γ_Σ) if $|C \cap \Gamma_\Sigma| = \#(C \cap \Gamma_\Sigma)$. Given a tight contact manifold $(T^2 \times I, \xi)$ satisfying Equation (1), we say an annulus $A \subset T^2 \times I$ is *horizontal* if A is a properly embedded, convex annulus with Legendrian boundary, and ∂A is efficient with respect to $\Gamma_{T_0} \cup \Gamma_{T_1}$. Horizontal annuli encode much (sometimes all) of the information on the tight contact manifold $(T^2 \times I, \xi)$.

Proposition 2.2. *Consider $(T^2 \times I, \xi_k)$ above with $\Gamma_{T_i} = 2l$, $i = 0, 1$. Let $A = S^1 \times \{pt\} \times I$ be a horizontal annulus which minimizes $\#\Gamma_A$ amongst horizontal annuli isotopic to A . Then Γ_A consists of $k - 1$ closed curves and l ∂ -parallel arcs abutting each boundary component of the annulus. Moreover, any other horizontal annulus A' isotopic to A will have $\Gamma_{A'}$ which can be obtained from Γ_A by adding $2m$ extra closed curves, $m \in \mathbb{Z}^{\geq 0}$. In particular, distinct ξ_k 's are distinguished by minimal horizontal annuli.*

Next, $(T^2 \times I, \xi)$ satisfying Equation (1) is called *rotative* if there is a convex torus T' parallel to T_0 for which $\text{slope}(\Gamma_{T'}) \neq \text{slope}(\Gamma_{T_0})$. $(T^2 \times I, \xi)$ which is not rotative is *non-rotative*. The following proposition explains the relationship between non-rotative ξ and their horizontal annuli A .

Proposition 2.3. *If ξ satisfies Equation (1) and A is a horizontal annulus, then ξ is non-rotative if and only if there exists a dividing curve of Γ_A which is nonseparating (i.e., connects the two boundary components of A). Moreover, for a non-rotative ξ , the isotopy class of the contact structure (rel boundary) is determined completely by Γ_A . In particular, ξ is contactomorphic to an S^1 -invariant contact structure on $S^1 \times A$, where $\{pt\} \times A$ has dividing set Γ_A .*

An important consequence of Proposition 2.3 is that non-rotative $(T^2 \times I, \xi)$ are in 1-1 correspondence with horizontal annuli A which are nonseparating. Now, according to Giroux's criterion [13] (also see [16]), a convex surface $\neq S^2$ has a tight neighborhood if and only if there is no homotopically trivial dividing curve. (Moreover, the tight neighborhood is automatically universally tight.) Therefore, two non-rotative, universally tight $(T^2 \times [0, 1], \xi_1)$ and $(T^2 \times [1, 2], \xi_2)$ glue into a universally tight contact structure if and only if their corresponding horizontal annuli A_1 and A_2 do not glue to give a homotopically trivial dividing curve.

On the other hand, any rotative ξ is isomorphic to a universally tight contact structure obtained by taking $(T^2 \times [0, 1], \xi_k)$ for some $k \in \mathbb{Z}^+$ and attaching non-rotative outer layers $T^2 \times [-1, 0]$ and $T^2 \times [1, 2]$.

Now, consider a contact manifold (M, ξ) and a torus $T \subset M$. The *torsion* of the isotopy class $[T]$ of T , due to Giroux [11], is defined as follows. Let $\text{tor}([T], M, \xi)$ be the supremum over $k \in \mathbb{Z}^{\geq 0}$, for which there exists a contact embedding $(T^2 \times I, \xi_k) \hookrightarrow (M, \xi)$ where the image of $T^2 \times \{pt\}$ is isotopic to T . The following proposition is a consequence of the work of Giroux [11] and Kanda [23].

Proposition 2.4. $\text{tor}([T_0], T^2 \times I, \xi_k) = k$.

2.2. The Attach = Dig Principle. Let $(T^2 \times [0, 1], \xi)$ be a rotative universally tight contact structure with $\#\Gamma_{T_1} > 2$. We explain the effect of attaching a bypass along T_1 from the outside along a Legendrian arc a which connects 3 different components of Γ_{T_1} . This operation reduces the number of components of Γ_{T_1} by two and corresponds to attaching a non-rotative $(T^2 \times [1, 2], \xi')$ layer. If we extend the attaching arc a of the bypass to a closed Legendrian curve α which is efficient with respect to Γ_{T_1} and intersects each component of Γ_{T_1} exactly once, the dividing set that ξ' induces on the horizontal annulus $A' = \alpha \times [1, 2]$ consists of an arc connecting two points on a lying between the dividing curves of T_1 , and nonseparating arcs emanating from all other midpoints of segments of $\alpha \setminus \Gamma_{T_1}$. Now we make the following simplifying assumption (which is satisfied by ξ_k):

∂ -Parallel Assumption. *The dividing set of ξ on a horizontal annulus $A = \alpha \times [0, 1]$ consists of ∂ -parallel arcs and possibly closed curves.*

Let us mark the components of Γ_{T_1} that are straddled by the ∂ -parallel arcs on A (by \times in Figure 2). Note that adding a bypass to ξ across any marked curve will produce an overtwisted structure, since the dividing set on the extended annulus will contain a closed trivial dividing curve. Any bypass we discuss here is along an arc connecting three different dividing curves, and we call it a bypass “across” the middle curve.

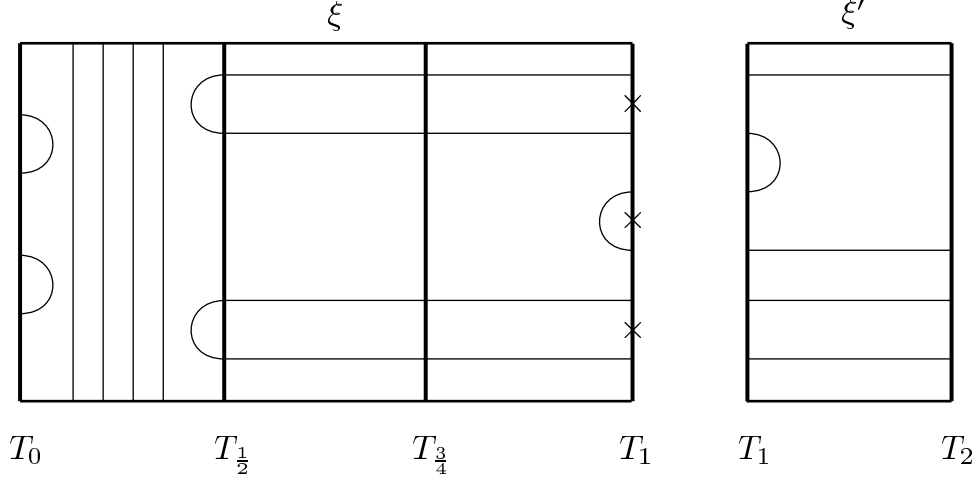


FIGURE 2. Templates. The top and bottom are identified.

Proposition 2.5. *Let $(T^2 \times [0, 1], \xi)$ be a rotative universally tight contact structure which satisfies $\#\Gamma_{T_1} \gg 2$ and the ∂ -Parallel Assumption, and let $(T^2 \times [1, 2], \xi')$ be a non-rotative layer corresponding to attaching a bypass across an unmarked dividing curve on T_1 . There exists a contact isotopy $\phi_t : (T^2 \times [0, 2], \xi \cup \xi') \hookrightarrow (T^2 \times [0, 2], \xi \cup \xi')$, $t \in [0, 1]$, rel T_0 , where $\phi_0 = id$ and $\phi_1(T^2 \times [0, 2]) \subset T^2 \times [0, 1]$.*

Proof. Observe that $(T^2 \times [0, 1], \xi)$ can be factored (after possibly isotoping relative to the boundary) into $(T^2 \times [0, \frac{1}{2}], \xi) \cup (T^2 \times [\frac{1}{2}, 1], \xi)$, where $T_{1/2}$ is convex, ξ is rotative on $T^2 \times [0, \frac{1}{2}]$, non-rotative on $T^2 \times [\frac{1}{2}, 1]$ and the dividing set induced on the annulus $A'' = \alpha \times [\frac{1}{2}, 1]$ consists of one ∂ -parallel arc and nonseparating arcs (see Figure 2). The factorization can be chosen so that the ∂ -parallel arc straddles one of the marked dividing curves adjacent to the unmarked curve across which the bypass of the layer $(T^2 \times [1, 2], \xi')$ is added. Note that $\#\Gamma_{T_{1/2}} = \#\Gamma_{T_1} - 2$. Finally, consider $(T^2 \times [\frac{1}{2}, 2], \xi \cup \xi')$. It is isotopic to a product neighborhood $\xi|_{T^2 \times [\frac{1}{2}, \frac{3}{4}]}$ of $T_{1/2}$ through an isotopy which fixes $T_{1/2}$. This is because a non-rotative contact structure is determined by the dividing set on a horizontal annulus (see Figure 3). \square

The same proof shows that digging out a single bypass along A (removing a portion like $T \times [\frac{1}{2}, 1]$ above) is equivalent to adding a bypass across an adjacent unmarked dividing curve. Therefore, we find that “attaching a $T^2 \times I$ layer” is the same as “digging out a $T^2 \times I$ layer”.

The Attach = Dig Principle applies more generally to a compact 3-manifold M with torus boundary T assuming $M = M' \cup_{\partial M' = T \times \{0\}} (T \times [0, 1])$, and ξ is rotative on $T \times [0, 1]$ with $\#\Gamma_{T \times \{1\}} \gg 2$ and satisfies the ∂ -Parallel Assumption. As in the torus case, we can use a horizontal annulus A for $T \times [0, 1]$ to mark dividing curves which are straddled by ∂ -parallel arcs in Γ_A . It follows from the previous proposition that attaching a bypass across an unmarked dividing curve is equivalent to a dig. It takes a bit more work to argue that digging out a bypass is equivalent to attaching one. Consider the factorization $M = M_B \cup (T \times [0, 1])$,

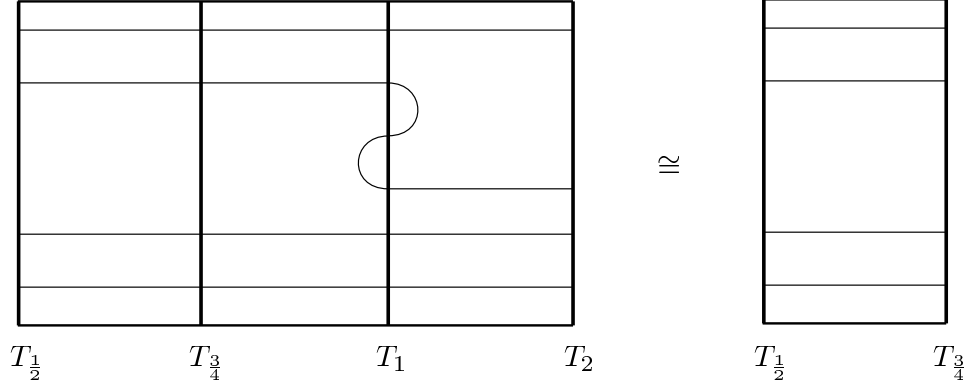


FIGURE 3.

where $T \times [0, 1]$ corresponds to a bypass attached to $T_1 \subset \partial M$ on the inside along an arc connecting three different dividing curves (a “short” bypass).

First of all, we know that any “short” bypass on the inside must be happening across a marked curve. Indeed if there was an inside bypass across an unmarked curve there would be an obvious overtwisted disc in the contact structure obtained by attaching the bypass on the outside across the same dividing curve. This would be a contradiction since we know that attaching such a bypass is equivalent to a dig and hence produces a tight structure (a subset of a tight structure is clearly tight). We can now look at the factorization $M = M_B \cup (T \times [0, 1])$ and conclude that (M_B, ξ) is isotopic to $(M_B \cup (T \times [0, 1]) \cup (T \times [1, 2]), \xi \cup \xi')$ with $(T \times [1, 2], \xi')$ an outside bypass across an adjacent unmarked curve.

The advantage of digging over attaching is manifest when we try to prove universal tightness by using the state transition method. We will also use the fact that digging can be viewed as attaching in proving a gluing theorem along incompressible tori.

2.3. Folding. We will now explain the process of *folding*, which plays a crucial role in our proof. A similar discussion can be found in [16]. Let Σ be a convex surface and γ be a *nonisolating* closed curve with $\gamma \cap \Gamma_\Sigma = \emptyset$. A closed curve γ is called *nonisolating* if every component of $\Sigma \setminus \gamma$ intersects Γ_Σ . Now, the Legendrian realization principle (see [16]) states that if X is a contact vector field transverse to Σ and γ is a nonisolating curve, then there is a C^0 -small isotopy ϕ_t , $t \in [0, 1]$, supported in a neighborhood of Σ , for which

1. $\phi_0 = id$,
2. $\forall t, \phi_t(\Sigma) \pitchfork X$,
3. $\forall t, \phi_t = id$ on Γ_Σ and $\Gamma_{\phi_t(\Sigma)} = \Gamma_\Sigma$,
4. $\phi_1(\gamma)$ is Legendrian.

Therefore, by using the Legendrian realization principle and possibly modifying Σ , we may take γ to be Legendrian curve. A slight strengthening of the Legendrian realization principle allows us to take γ to be a *Legendrian divide*. This means that all the points of γ are tangencies, and there exists a local model $N = S^1 \times [-\varepsilon, \varepsilon] \times [-1, 1]$ with coordinates (θ, y, z) and 1-form $\alpha = dz - yd\theta$ such that $\Sigma \cap N = S^1 \times [-\varepsilon, \varepsilon] \times \{0\}$ and $\gamma = S^1 \times \{0\} \times \{0\}$. A *fold* is a modification $\Sigma \mapsto \Sigma'$ where Σ and Σ' are isotopic and $\Sigma = \Sigma'$ outside N . Inside N , we bend Σ into S^1 times an S-shape so that Γ'_Σ consists of Γ_Σ outside N , together with two

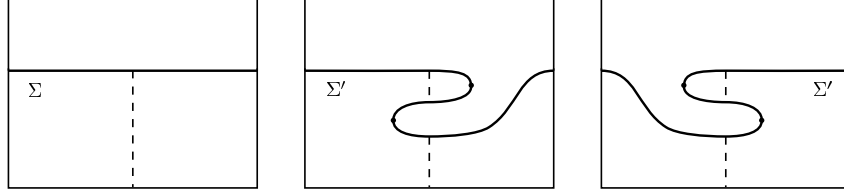


FIGURE 4. Folding.

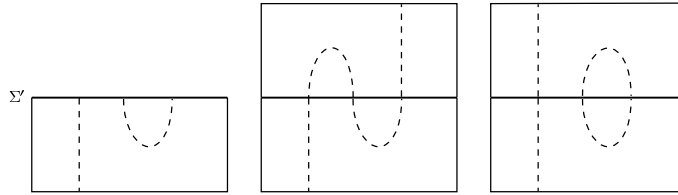


FIGURE 5. Templates.

new parallel curves isotopic to γ . The picture on the left-hand side of Figure 4 represents Σ . Here the rectangle represents $[-\varepsilon, \varepsilon] \times [-1, 1]$ (the horizontal direction is the y -direction and the vertical direction is the z -direction). There are two choices for folding (middle and right pictures), and they are not contact isotopic (see next paragraph). The dots (times S^1) represent the new dividing curves on Σ' . Note that if we let N' be the “upper” solid torus split by Σ' , the dividing curves of the meridional disks (after Legendrian realization to make the boundary of the meridional disks Legendrian) will be given by the dotted lines in Figure 4. Observe that we folded Σ to obtain Σ' inside an I -invariant neighborhood of Σ . Hence, folding to increase the number of dividing curves (in pairs) is an operation that can be done to change Γ_Σ *for free* (at least for nonisolating curves). In general, there are no other operations (such as decreasing the number of dividing curves) which can be performed on the dividing sets without prior knowledge of the ambient manifold or at least more global information.

To distinguish the tight contact structures obtained by folding, we apply the *template method*, which is similar to the method used in Section 2.2 to identify the location of the ∂ -parallel arcs. Namely, we glue on various $S^1 \times D^2$ with S^1 -invariant universally tight structures, and distinguish the folds according to which templates make the glued-up contact structure overtwisted. For the single fold above, we attach templates depicted in Figure 5. The top rectangles of the middle and right diagrams are the meridians of the templates. Note that the middle attachment preserves tightness, but the right attachment gives an overtwisted contact structure. On the other hand, if we took the mirror image of the left-hand diagram (across a vertical line), then the middle attachment will be overtwisted and the right attachment will be universally tight. In a similar manner, all the folds are distinguishable by attaching templates *from the outside*.

3. TORUS BOUNDARY CASE

In this section we will first construct a suitable tight contact structure ξ on M by using Theorem 1.6. Consider the situation where $\partial M \neq \emptyset$ and $\partial M = \cup_{i=1}^l T_i$, where T_i are incompressible tori. We choose a suitable surface S as the first splitting surface in a well-groomed sutured decomposition.

Claim. *There exists a properly embedded, not necessarily connected, well-groomed surface S which is (Thurston) norm-minimizing in $H_2(M, \partial M)$ and nontrivially intersects each T_i , that is, the image under the boundary map is a nonzero class in $H_1(T_i)$.*

Proof. For each $i = 1, \dots, l$, take $M(i)$ to be M filled with solid tori $S^1 \times D^2$ along each T_j except for T_i . This means $\partial M(i) = T_i$. Using the relative homology sequence and Lefschetz duality for the manifold $M(i)$ and its boundary $\partial M(i)$, we find a class $[S(i)] \in H_2(M(i), \partial M(i))$ which is nonzero under the boundary map $H_2(M(i), \partial M(i)) \xrightarrow{\partial} H_1(M(i))$. Represent it by a surface $S(i)$ in $M(i)$. Back on $M \subset M(i)$ let $S_i = S(i) \cap M$ and consider $[S(i) \cap T_j]$ for $j \neq i$ (assume transversality). If $[S_i \cap T_j] = 0 \in H_1(T_j)$ we can pair off the intersections and get $S_i \cap T_j = \emptyset$. Now, take a suitable linear combination of the $[S_i]$ to obtain $[S] \in H_2(M, \partial M)$ for which $\partial[S]$ is nonzero when restricted to each $H_1(T_i)$. Finally pick a norm-minimizing representative of $[S]$. The well-grooming is simply asking that $T_i \cap \partial S$ be parallel curves oriented in the same direction. (If not already well-groomed, we can pair off oppositely oriented parallel curves without altering the norm.) \square

To proceed with the construction, let $\Gamma_{\partial M}$ consist of pairs of parallel essential curves for each T_i , chosen so that Γ_{T_i} has nonzero intersection with $T_i \cap \partial S$ on T_i and that every component of $T_i \cap \partial S$ is efficient with respect to $\Gamma_{\partial M}$. $\Gamma_{\partial M}$ is taut, since each component of R_{\pm} is an annulus. The order of choosing things here might seem reverse from the way we study contact structures by decomposing them along convex surfaces, and it is — we are choosing the norm-minimizing surface S first, and then choosing $\Gamma_{\partial M}$. We are free to do that since we are constructing a contact structure by gluing, rather than analyzing one that is already given to us. Let us assume first that all the T_i in the sutured decomposition are torus sutures. We have chosen the next cutting surface S to be norm-minimizing and well-groomed, implying that the resulting sutured manifold $M \setminus S$ is taut. Now, choosing $\Gamma_{\partial M}$ to have nonzero geometric intersection with each T_i , together with the well-grooming of S , guarantees that after splitting and rounding, the sutures (dividing curves) on $\Gamma_{\partial(M \setminus S)}$ are the same as would have been obtained from the sutured decomposition in the case when all the T_i are torus sutures. Theorem 1.3 ensures a sutured manifold hierarchy where the first splitting is along S . Note that Gabai's theorem also ensures that each subsequent cutting is done along surfaces which are well-groomed and have boundary. Therefore, by Theorem 1.4, we can construct a universally tight contact structure ξ on M for which S is convex with Γ_S ∂ -parallel.

Assume that we have fixed $\Gamma_{\partial M}$ as above and constructed a specific universally tight (M, ξ) . We will proceed to describe a family of contact structures ξ'_k , $k \in \mathbb{Z}^+$, obtained by attaching $(T^2 \times I, \xi_k)$ layers onto M along $T = T_i$, where ξ_k is defined as in Section 2.1. Write $\xi_k^+ = \xi_k$, $k \in \mathbb{Z}^+$, and let ξ_k^- be given by $-\alpha_k$ (the 2-plane fields ξ_k^+ and ξ_k^- are identical, but are oriented oppositely). Consider the glued-up contact manifold $(M' = M \cup_T (T^2 \times I), \xi'_k)$,

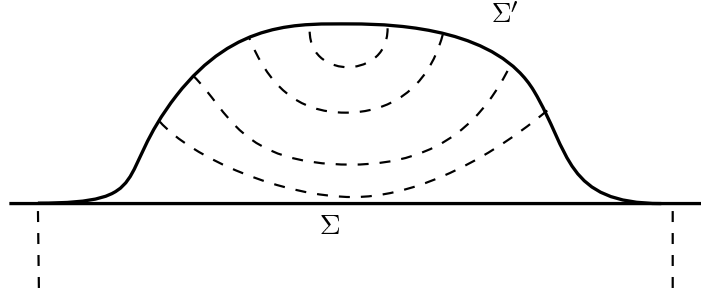


FIGURE 6. Meridian.

where we use some element of $SL(2, \mathbb{Z})$ to glue $T^2 \times \{0\}$ to T so that the dividing curves match up.

Proposition 3.1. ξ'_k is universally tight for precisely one of ξ_k^+ or ξ_k^- .

Proof. Cut M' open along $S' = S \cup ((\partial S \cap T) \times I)$, where $T = T \times \{0\}$. Denote the components of ∂S by $\gamma_1, \dots, \gamma_s, \gamma_{s+1}, \dots, \gamma_t$, where $\gamma_i, i = 1, \dots, s$, are parallel curves on T and the other γ_i lie on different $T_j \neq T$. Then $S' \setminus S$ consists of annuli $\gamma_i \times I, i = 1, \dots, t$. Let A_1, \dots, A_s be the annular components of $(T \times \{1\}) \setminus \partial S'$ and A_{s+1}, \dots, A_t be the annular components of $\cup_{T_j \neq T} (T_j \times \{1\}) \setminus \partial S'$. Recall that S was constructed so that $T_j \cap S \neq \emptyset$ for all T_j . By possibly reordering, we may assume that $\partial A_i = (\gamma_{i-1} \times \{1\}) \cup (\gamma_i \times \{1\}), i = 1, \dots, s$, where $\gamma_0 \stackrel{\text{def}}{=} \gamma_s$.

First observe that Proposition 2.2 implies that, for a correct choice of ξ_k^+ or ξ_k^- (but not both, since in one case the ∂ -parallel components will match to form a closed curve parallel to the boundary, and in the other to form homotopically trivial circles, giving overtwisted disks) $\Gamma_{S'}$ will consist of ∂ -parallel curves identical to those of Γ_S , together with k extra curves parallel to γ_i for each $i = 1, \dots, s$. $\Gamma_{\partial(M \setminus S)}$ is isotopic to the collection of curves which consists of a single core curve from each $A_i, i = 1, \dots, t$; $\Gamma_{\partial(M' \setminus S')}$ is $\Gamma_{\partial(M \setminus S)}$ with $2k$ extra curves parallel to the core curve of A_i , for each $i = 1, \dots, s$.

Claim. If every connected component of S' has at least two boundary components, then the contact structure ξ'_k on $M' \setminus S'$ is the tight contact structure ξ on $M \setminus S$ with folds introduced.

Proof of claim. Observe that $M' \setminus S' = (M \setminus S) \cup ((T \setminus (\cup_{i=1}^s \gamma_i)) \times I)$. Let $\gamma'_i, i = 1, \dots, s$ be parallel copies of γ_i on S which do not intersect Γ_S , and let $B_i \subset S'$ be the annulus bounded by γ'_i and $\gamma_i \times \{1\}$. Then take the annulus $\Sigma' \subset \partial(M' \setminus S')$ to be a union of A_i and copies of B_{i-1} and B_i . Also let $\Sigma \subset \partial(M \setminus S)$ be an annulus parallel to Σ' with γ'_i, γ'_{i+1} as boundary. We may use the Legendrian realization principle to realize copies of γ'_i and γ'_{i+1} on Σ (and Σ') as Legendrian divides. It is at this point that the assumption that every connected component of S' has at least two boundary components is needed to conclude that the nonisolating condition in the Legendrian realization principle is met. Now, after rounding, Σ and Σ' are disjoint except along their common boundary $\gamma'_i \cup \gamma'_{i+1}$. Let N be the solid torus region bounded by Σ and Σ' . The dividing curves on the meridian of N will be as in Figure 6. With this explicit description, it is immediate that the attachment of

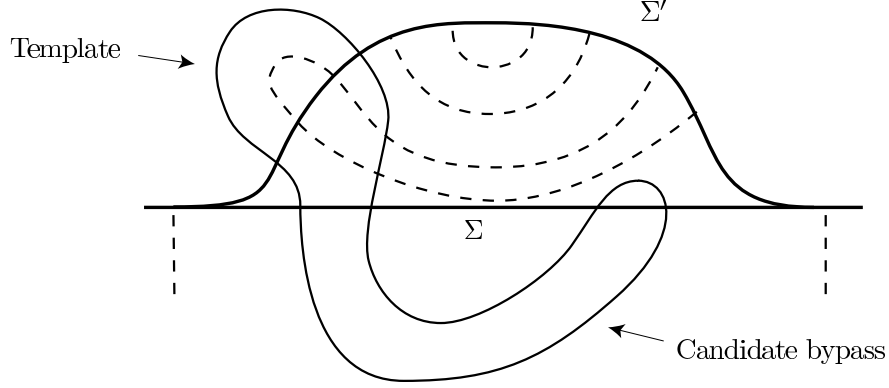


FIGURE 7. Attaching a template to obtain an overtwisted disk.

N is equivalent to a folding operation. Finally, repeat this procedure with all annuli A_i , $i = 1, \dots, s$, in succession. \square

The assumption of the previous claim on the number of boundary components is a technical condition. It is satisfied by passing to a large finite cover \widetilde{M}' which unwinds S' to give \widetilde{S}' , which would have more than one boundary component even if S' had one boundary component. The claim implies that there exists a contact embedding:

$$(\widetilde{M}' \setminus \widetilde{S}', \widetilde{\xi}'_k) \hookrightarrow (\widetilde{M} \setminus \widetilde{S}, \widetilde{\xi}).$$

Therefore, ξ'_k on $M' \setminus S'$ is universally tight.

Next, we need to deal with the problem of gluing M' back along S' . The argument is almost identical to the proof of the Gluing Theorem (Theorem 1.6) in the ∂ -parallel case. Pass to a large finite cover \widetilde{M} of M , so that we have a large cover \widetilde{S}' of S' , and lift the candidate overtwisted disk D . Here, every bypass attachment will be either trivial or will decrease the number of dividing curves. A bypass along \widetilde{S}' which decreases the number of dividing curves cannot exist by attaching a template. See Figure 7. Therefore, (M', ξ'_k) is universally tight. \square

Remark. The above proposition could have been proved using Colin's Gluing Theorem (Theorem 4.6) below. Our point was to demonstrate a different perspective.

Change of notation. To avoid excessive use of primes, from now on we will refer to the contact manifold (M', ξ'_k) and the splitting surface S' as (M, ζ_k) and S , respectively. Also write $\partial M = \cup_{i=1}^l T_i$

Proposition 3.2. *The universally tight contact structures ζ_k are nonisotopic rel boundary.*

Proof. Assume without loss of generality that S has collared Legendrian boundary. Let T be the connected component of ∂M in whose direction twisting was added. Now let $\delta : [0, 1] \rightarrow S$ be a properly embedded arc on S satisfying the following:

- (a) δ is essential (i.e., not ∂ -compressible).
- (b) The endpoints $\delta(0), \delta(1)$ lie on the *Legendrian divides* of the collared Legendrian boundary of S .

- (c) At least one of $\delta(0)$, $\delta(1)$ (both if possible) lies on T .
- (d) δ has minimal geometric intersection $|\delta \cap \Gamma_S|$ amongst arcs on S isotopic to δ with endpoints on the same component of ∂S as δ .
- (e) δ is Legendrian. (This is made possible by the Legendrian realization principle.)

Since Γ_S consists of ∂ -parallel arcs and closed curves parallel to ∂S , any choice of δ will have no intersections with the ∂ -parallel dividing arcs and minimally intersect the closed dividing curves parallel to the boundary, i.e., k times if only one of $\delta(0)$, $\delta(1)$ lie on T , and $2k$ times if they both do. Define \mathcal{L}_δ to be the set of Legendrian arcs d satisfying the following:

1. There exists a properly embedded convex surface $S' \supset d$ with collared Legendrian boundary.
2. $\partial S'$ is efficient with respect to $\Gamma_{\partial M}$.
3. The endpoints of d lie on Legendrian divides of the collared Legendrian boundary of S' .
4. There exists an isotopy (not necessarily a contact isotopy) which sends S to S' and δ to d , where ∂S is isotoped to $\partial S'$ along ∂M .

We are now ready to define our invariant which distinguishes the ζ_k . For any $d \in \mathcal{L}_\delta$ we define the *twisting number* $t(d)$ to be $-\frac{1}{2}|d \cap \Gamma_{S'}|$, where S' is the convex surface containing d in the definition of \mathcal{L}_δ above. Define the *maximum twisting* for \mathcal{L}_δ to be

$$t(\mathcal{L}_\delta) = \max_{d \in \mathcal{L}_\delta} t(d).$$

We claim that $t(\mathcal{L}_\delta) = t(\delta)$. Assume on the contrary that there exists a Legendrian arc $d \in \mathcal{L}_\delta$ with $t(d) = t(\delta) + 1$, sitting on the convex surface S' . There exists an isotopy $\phi_t : [0, 1] \rightarrow M$, $t \in [0, 1]$, such that $\phi_0 = \delta$ and $\phi_1 = d$. Since ∂S and $\partial S'$ are efficient with respect to $\Gamma_{\partial M}$ and are isotopic on ∂M through efficient curves, it is possible to “slide” S along ∂M without altering the isotopy class of Γ_S , and we may assume that $\partial S = \partial S'$ and $\phi_t(0)$, $\phi_t(1)$ are fixed. As usual, pass to a large finite cover \widetilde{M} of M to unwrap S to \widetilde{S} .

Claim. *For a large enough finite cover \widetilde{M} , there exists a lift/extension of $\phi_t : [0, 1] \rightarrow M$, $t \in [0, 1]$, to $\Phi_t : \widetilde{S} \rightarrow \widetilde{M}$, where Φ_0 is the identity and the support of the isotopy Φ_t is compact and is contained inside an embedded 3-ball.*

Proof of claim. The key property we use here is that the isotopy ϕ_t , $t \in [0, 1]$, can be lifted to a large finite cover \widetilde{M} so that the trace of the lift $\widetilde{\phi}_t : [0, 1] \rightarrow \widetilde{M}$ is contained in a 3-ball B^3 . This follows from the fact that M is Haken, which implies that M has universal cover \mathbb{R}^3 and $\pi_1(M)$ is residually finite. Let \widetilde{X}_t , $t \in [0, 1]$, be a time-dependent vector field along $\widetilde{\phi}_t([0, 1])$ which induces the isotopy $\widetilde{\phi}_t$. Damp \widetilde{X}_t out away from $\widetilde{\phi}_t([0, 1])$ and extend it to all of \widetilde{M} . Then $\cup_{t \in [0, 1]} \text{supp}(\widetilde{X}_t) \subset B^3$, where the $\text{supp}(X)$ is the support of a vector field (i.e., $p \in M$ such that $X(p) \neq 0$). Let Φ_t , $t \in [0, 1]$, be the flow corresponding to \widetilde{X}_t . It is supported inside B^3 from our construction. \square

By assumption, $|t(d)|$ is strictly smaller than k (or $2k$). If we can prove that, for each component C of $\partial S'$ which is on T , there exist at least k closed curves in $\Gamma_{S'}$ parallel to C , we have a contradiction. In fact it is clearly enough to obtain the contradiction for any large enough cover. Take the cover \widetilde{M} of M given by the previous claim, and use the technique

of *isotopy discretization*, which first appeared in Colin [2], and is a method which works extremely well in the context of convex surfaces and state transitions (due to bypasses). We split $[0, 1]$ into small time intervals $t_0 = 0 < t_1 < \dots < t_{n-1} < t_n = 1$, $n \gg 0$, so that all the Φ_t are all “sufficiently close” if $t \in [t_i, t_{i+1}]$, in the following sense: there exists an embedding $\psi_i : \tilde{S} \times I \rightarrow \tilde{M}$ such that $\Phi_t(\tilde{S}) \subset \psi_i(\tilde{S} \times I)$ and $\pi_1 \circ \psi_i^{-1} \circ \Phi_t(\tilde{S})$ is a submersion for all $t \in [t_i, t_{i+1}]$. (Here $\pi_1 : \tilde{S} \times I \rightarrow \tilde{S}$ is the first projection.) This implies (by Giroux’s convex movie approach in [12] and the equivalence of Giroux’s ‘retrogradient switch’ with the notion of a bypass) that to get from $\Phi_0(\tilde{S}) = \tilde{S}$ to $\Phi_1(\tilde{S})$ we move via a sequence of bypass moves. We may need to take an even larger cover of \tilde{M} to expand \tilde{S} (without changing names) and extend B^3 so that we can “localize” all the bypasses, i.e., all the allowable bypasses are either trivial or reduce the number of dividing curves. The bypasses which potentially give us trouble are the bypasses which increase the number of dividing curves. Such a bypass occurs only when the bypass “wraps around” a closed curve which is homotopically nontrivial — by unwrapping in this direction in a large enough cover, we may avoid a dividing curve increase. More precisely, since S is incompressible in M , $\pi_1(S)$ injects into $\pi_1(M)$, and we may unwrap S in the direction of any homotopically nontrivial curve in a finite cover of M using residual finiteness.

We can show that there are no bypasses reducing the number of dividing curves as in the proof of Proposition 3.1. Therefore, we keep attaching trivial bypasses, and eventually find a contact isotopy of \tilde{S} with $\Phi_1(\tilde{S})$, which contradicts the assumption that $t(d) > t(\delta)$. \square

More generally, let ζ_{k_1, \dots, k_l} be the universally tight contact structure obtained by attaching appropriate $(T^2 \times I, \xi_{k_i}^\pm)$ layers ($i = 1, \dots, l$) to each component T_i of ∂M .

Proposition 3.3. *The universally tight contact structures ζ_{k_1, \dots, k_l} are distinct up to isotopy rel boundary.*

4. CLOSED CASE

4.1. Topological preliminaries.

4.1.1. *Residual finiteness.* A group G is said to be *residually finite* if given any nontrivial element $g \in G$ there exists a finite index *normal* subgroup $H \subset G$ which does not contain g . When G is the fundamental group $\pi_1(M)$ of a manifold M , the residual finiteness of G is equivalent to the following: If K is a compact subset of the universal cover \overline{M} of M , then there exists a finite cover \tilde{M} of M for which the projection $\overline{M} \rightarrow \tilde{M}$ is injective on K . Note that since $\pi_1(\tilde{M})$ can be taken to be *normal* in $\pi_1(M)$, \tilde{M} is a *Galois* (or *regular*) cover. We list some facts about residual finiteness (which can be found in [1]):

1. A subgroup of a residually finite group G is residually finite.
2. (Hempel [15]) If M is Haken, then $\pi_1(M)$ is residually finite.

Let H be a subgroup of a group G . We say H is *separable* in G if for any element $g \in G - H$ there exists a finite index subgroup K in G satisfying $H \subset K$ and $g \notin K$. The following is a result of Long and Niblo [24]:

Theorem 4.1. *Let M be a closed oriented Haken 3-manifold. If $i : T^2 \rightarrow M$ is an incompressible embedded torus, then $i_*(\pi_1(T^2))$ is a separable subgroup in $\pi_1(M)$.*

Let M be a compact manifold with an incompressible torus T . We explain what it means to *expand* T and obtain a finite cover \widetilde{M} of M . By property (1) above, if we choose a finite collection \mathcal{C} of elements of $\pi_1(T) = \mathbb{Z}^2$, there exists a finite index normal subgroup H which avoids \mathcal{C} . If $\pi_1(T)$ has basis α, β , and we take $\mathcal{C} = \{m\alpha + n\beta \mid 0 < m^2 + n^2 \leq R\}$ for some large $R > 0$, we obtain a finite Galois cover \widetilde{M} of M which *expands* T .

4.1.2. Torus decompositions. The following is the Torus Decomposition Theorem (also often called the JSJ decomposition), due to Jaco-Shalen [21] and Johansson [22].

Theorem 4.2. *Let M be a closed, oriented, irreducible 3-manifold. Then, up to isotopy, there exists a unique collection \mathcal{T} of disjoint incompressible tori T_1, \dots, T_m which satisfies the following:*

1. *Each component of $M \setminus \bigcup_{i=1}^m T_i$ is atoroidal (i.e., any incompressible torus can be isotoped into a boundary torus), or else is Seifert fibered.*
2. *\mathcal{T} is minimal in the sense that (1) fails when any T_i is removed.*

Most atoroidal components can be endowed with a hyperbolic structure with cusps, except for the following potential components which are Seifert fibered: (1) Seifert fibered spaces with m singular fibers over a base Σ which is S^2 with n punctures, where $m + n \leq 3$. (2) Seifert fibered space with m singular fibers over a base Σ which is \mathbb{RP}^2 with n punctures, where $m + n \leq 2$.

A clear account of the Torus Decomposition Theorem appears in [14]. The following facts, also found in [14] are useful.

Proposition 4.3. *Let M be a connected, compact, oriented, irreducible Seifert fibered space. Then any incompressible, ∂ -incompressible properly embedded surface $S \subset M$ is isotopic to a surface which is either vertical (union of regular fibers), or horizontal (transverse to all fibers).*

Lemma 4.4. *Let M be a connected, compact, oriented, irreducible, and atoroidal. If M contains an incompressible, ∂ -incompressible annulus meeting only torus components of ∂M , then M is a Seifert fibered space.*

4.1.3. Diffeomorphism groups of 3-manifolds. We also need to recall the following fact regarding the mapping class group $\pi_0(\text{Diff}(M))$ of a Haken 3-manifold M .

Theorem 4.5. *Let M be an oriented, compact Haken 3-manifold and $\text{Diff}(M)$ its group of diffeomorphisms. Then $\pi_0(\text{Diff}(M))$ is finite modulo Dehn twisting along possible incompressible tori and incompressible, ∂ -incompressible annuli.*

For a 3-manifold, a *Dehn twist* is a diffeomorphism ϕ which is locally given on $T^2 \times I = (\mathbb{R}^2/\mathbb{Z}^2) \times [0, 1]$ or $[0, 1] \times (\mathbb{R}/\mathbb{Z}) \times [0, 1]$ with coordinates (x, y, t) by: $\phi : (x, y, t) \mapsto (x, y + t, t)$.

4.2. Gluing along incompressible tori. The following result, due to Colin [3], gives a sufficient condition for gluing tight contact manifolds along tori:

Theorem 4.6 (Colin). *Let M be an oriented, irreducible 3-manifold and $T \subset M$ an incompressible torus. Consider a contact structure ξ for which T is pre-Lagrangian (linearly foliated) and $\xi|_{M \setminus T}$ is universally tight. Then ξ is universally tight on M .*

From the perspective of convex decomposition theory, this is wholly unsatisfactory, since the gluing surface is not convex! We therefore formulate a more cumbersome variant of Colin's Gluing Theorem which is phrased and proved entirely using convex surface theory.

Theorem 4.7 (Variant of Colin's Theorem). *Let M be an oriented, irreducible 3-manifold and $T \subset M$ an incompressible torus. Consider a contact structure ξ for which T is convex, ξ is universally tight when restricted to $M \setminus T$, there exists a toric annulus layer $N = T \times [-1, 1] \subset M$, where $T = T \times \{0\}$, $\xi|_N$ is universally tight and $\xi|_{T \times [-1, 0]}$ and $\xi|_{T \times [0, 1]}$ are both rotative. Then ξ is universally tight on M .*

Proof. Recall that, according to the classification of tight contact structures on toric annuli (see [16]), the universal tightness of $\xi|_N$ simply means that it is obtained by taking the standard rotative contact structure $\sin(2\pi z)dx + \cos(2\pi z)dy = 0$ on $T^2 \times \mathbb{R}$ with coordinates $((x, y), z)$, truncating for a certain interval $[z_0, z_1] \subset \mathbb{R}$, making the boundary convex with 2 dividing curves each, and finally folding to attach a non-rotative layer. Without loss of generality, we may take both $\xi|_{T \times [-1, 0]}$ and $\xi|_{T \times [0, 1]}$ to satisfy the ∂ -Parallel Assumption with ∂ -parallel arcs along T_0 .

We will prove that (M, ξ) is tight by contradiction. The proof of tightness of finite covers of M is identical, since any finite cover of (M, ξ) also satisfies conditions of Theorem 4.7. This implies the universal tightness of ξ , since Haken 3-manifolds have residually finite fundamental group, so any overtwisted disk that would exist in the universal cover could be 1-1 projected into a finite cover.

Assume (M, ξ) is overtwisted with an overtwisted disk $D \subset M$. As before, we look at a large finite cover $\pi : \widetilde{M} \rightarrow M$ that sufficiently expands T . Let \widetilde{T}_i be connected components of $\pi^{-1}(T)$ and \widetilde{D} be a lift of D to \widetilde{M} . In \widetilde{M} we may *extricate* \widetilde{D} from \widetilde{T}_i essentially without any penalty, as we shall see. As in the proof of Theorem 1.6, we push \widetilde{D} across \widetilde{T}_i in stages. We can eliminate components of $\widetilde{D} \setminus (\cup_i \widetilde{T}_i)$ by pushing across balls, and this is equivalent to a sequence of bypass moves. If we expand T sufficiently, i.e., we take a large enough cover, the only bypass operations are either (1) trivial or (2) reduce $\#\Gamma_{\widetilde{T}_i}$. In particular, there are no dividing curve increases or changes in slope.

Let \widetilde{T} be a fixed \widetilde{T}_i . Denote by $\widetilde{T} \times [-1, 1]$ the connected component of $\pi^{-1}(T \times [-1, 1])$ where $\widetilde{T} = \widetilde{T} \times \{0\}$. Suppose the bypass attached onto \widetilde{T} from the $\widetilde{T} \times [0, 1]$ side decreases $\#\Gamma_{\widetilde{T}}$. Let \widetilde{T}' be the new torus we obtain after the bypass attachment. Then $(\widetilde{M} \setminus \widetilde{T}', \pi^*(\xi))$ is tight by the Attach = Dig Principle (Section 2.2). Repeated application of the Attach = Dig Principle proves Theorem 4.7. \square

4.3. Proof of Theorem 0.2. Let M be a closed manifold and $T \subset M$ an oriented incompressible torus. The cut-open manifold $M \setminus T$ will have two torus boundary components which we denote by T_+ and T_- ; T_+ is the component where the boundary orientation agrees with the orientation induced from T , and T_- is the component where they are opposite. T can be either separating or nonseparating. If T is separating, then $M \setminus T$ will consist of two components M_1, M_2 , with $\partial M_1 = T_+$, $\partial M_2 = -T_-$. If T is nonseparating, $M \setminus T$ will consist of one component M_1 with $\partial M_1 = T_+ - T_-$. In either case, initially choose $\Gamma_{\partial(M \setminus T)}$ with $\Gamma_{T_+} = \Gamma_{T_-}$ so that splitting surfaces S_i , chosen below, intersect T_+ in a collection of parallel essential curves which are not parallel to Γ_T (and likewise for T_-).

Construct universally tight contact structures ζ_{k_+, k_-} on $M \setminus T$ as in Section 3 by (1) performing a convex decomposition corresponding to a sutured manifold decomposition and regluing as in the proof of Theorem 0.1 and (2) adding a πk_+ -twisting layer to T_+ and a πk_- -twisting layer to T_- as in Proposition 3.1. For the correct parity of $k = k_+ + k_-$, Theorem 4.7 gives a universally tight contact structure on M which we call ζ_k .

Theorem 4.8. *If M is a closed, oriented 3-manifold with an incompressible torus, then there exist infinitely many universally tight contact structures up to isotopy.*

Proof. We define the invariant which will allow us to distinguish the ζ_k up to isotopy. First we choose a suitable isotopy class of closed curves. If T separates, then take the next splitting surfaces S_i of M_i , $i = 1, 2$. Since S_i cannot be a disk with boundary on T , we may take arcs $\gamma_i \subset S_i$ with endpoints on T which are not ∂ -compressible on S_i as well as in M_i . Glue γ_1 and γ_2 to obtain γ . If T is nonseparating and there is an (incompressible) splitting surface S which spans between T_+ and T_- , then take γ_1 to be some arc on S with endpoints on T_+ and T_- and glue the endpoints to obtain γ . If there is no single surface S , then take S_1 to have boundary component(s) on T_+ and S_2 to have boundary components on T_- , find γ_i , $i = 1, 2$, as before, and glue to get γ . Let \mathcal{L}_γ be the set of Legendrian curves which are isotopic to γ (but not necessarily Legendrian isotopic).

Assume without loss of generality that we have two surfaces S_1 and S_2 . Recall ∂S_i , $i = 1, 2$, consist of efficient parallel oriented curves on T which have nonzero geometric intersection with Γ_T . We construct a surface $S_1 + S_2$ by performing a band sum at each (geometric) intersection of ∂S_1 and ∂S_2 . Whether we do a positive band sum or a negative band sum depends on the following. In order to define the band sum, temporarily assume Γ_T , ∂S_1 , ∂S_2 are all linear on T , with slopes ∞ , s_1 , s_2 , respectively. (That is, for the time being, forget the fact that ∂S_1 and ∂S_2 are Legendrian, and treat them just as curves on T .) Thicken T to $T \times [-\varepsilon, \varepsilon]$ so that $\partial S_1 \subset T \times \{\varepsilon\}$ and $\partial S_2 \subset T \times \{-\varepsilon\}$. Each band intersects $T \times \{t\}$, $t \in [-\varepsilon, \varepsilon]$, in a linear arc of slope $s(t)$, which interpolates between s_1 and s_2 and is never ∞ . In the case $s_1 = s_2$, take suitable multiples of S_1 and S_2 (still call them S_1 and S_2) so $\partial S_1 = \partial S_2$, and call this $S_1 + S_2$.

We now define the framing for γ to be one which comes from $S_1 + S_2$. Here we are assuming without loss of generality that γ_i , $i = 1, 2$, have endpoints on the same band. A natural Legendrian representative of γ will have subarcs γ_i , $i = 1, 2$, where the endpoints of γ_i lie on half-elliptic points on ∂S_i , which are also on Γ_T . This means that the actual Legendrian curves ∂S_1 and ∂S_2 intersect in a tangency at the endpoints of γ_i . As before, define the *maximal twisting* $t(\mathcal{L}_\gamma)$ to be the maximum twisting number of Legendrian curves in \mathcal{L}_γ . The following proposition proves Theorem 4.8. \square

Proposition 4.9. $t(\mathcal{L}_\gamma) = -\frac{1}{2}(k+1)$ or $-(k+1)$, depending on whether γ intersects T once or twice.

Proof. It is clear that $t(\mathcal{L}_\gamma) \geq -\frac{1}{2}(k+1)$ or $-(k+1)$, for these bounds are achieved by γ on $S_1 + S_2$. We need to show that $t(\mathcal{L}_\gamma) \leq -\frac{1}{2}(k+1)$ or $-(k+1)$. We will assume that there exist two surfaces S_1 and S_2 — the other case is similar. Suppose γ' is a Legendrian curve in \mathcal{L}_γ with $t(\gamma') = -(k+1) + 1$. There exists an isotopy $\phi_t : S^1 \rightarrow M$, $t \in [0, 1]$, for which

$\phi_0(S^1) = \gamma$ and $\phi_1(S^1) = \gamma'$. We pass to a large cover of T to remove extra intersections of γ' with T .

In order to reduce from the case where M is a closed manifold with incompressible torus T to the manifold-with-torus-boundary case, we construct an auxiliary contact manifold $(W, \hat{\zeta})$ which has the following properties:

1. W is a large finite cover of M (not necessarily Galois) and $\hat{\zeta}$ the pullback of ζ_k .
2. $\phi_t : S^1 \rightarrow M$, $t \in [0, 1]$, lift to closed curves $\Phi_t : S^1 \rightarrow W$. In other words, $\Phi_t(S^1)$ maps 1-1 down to $\phi_t(S^1)$ under the covering projection $\rho : W \rightarrow M$.
3. T has been expanded sufficiently on W so that the intersection of $\cup_{t \in [0, 1]} \Phi_t(S^1)$ with any component \hat{T}_i of $\rho^{-1}(T)$ is (either empty or) contained inside a “small” disk $D_i \subset \hat{T}_i$ which does not intersect most of the dividing curves of \hat{T}_i .
4. If we denote $\hat{\gamma} = \Phi_0(S^1)$ and $\hat{\gamma}' = \Phi_1(S^1)$, then $t(\gamma) = t(\hat{\gamma})$ and $t(\gamma') = t(\hat{\gamma}')$, where $t(\hat{\gamma})$ and $t(\hat{\gamma}')$ are measured with respect to preimages \hat{S}_1 and \hat{S}_2 of S_1 and S_2 .

The construction of W will be done in the next section. In the meantime we continue with the proof of Proposition 4.9, assuming W has been constructed. Φ_t gives rise to a time-dependent vector field X_t along $\Phi_t(S^1)$; extend X_t and hence Φ_t to all of W by damping out outside a small neighborhood of $\Phi_t(S^1)$. Then the support $\text{supp}(X_t)$ of X_t is contained in $N(\Phi_t(S^1))$. For each \hat{T}_i , $\Phi_t(\hat{T}_i)$ remains constant outside a disk $D_i \subset \hat{T}_i$ which we may take to be convex with Legendrian boundary. If we discretize this isotopy into small time intervals $t_0 = 0 < t_1 < \dots < t_n = 1$, then each step $\Phi_{t_k}(\hat{T}_i)$ to $\Phi_{t_{k+1}}(\hat{T}_i)$ corresponds to attaching a bypass which either is trivial or decreases the number of dividing curves while keeping the slope unchanged. Using the “attach = dig” principle, we find that the contact manifold $(W \setminus (\cup_i \Phi_1(\hat{T}_i)), \hat{\zeta})$ is obtained from $(W \setminus (\cup_i \hat{T}_i), \hat{\zeta})$ by attaching dividing-curve-decreasing bypasses or “unfolding”. Observe that $\hat{\gamma}'$ intersects each $\Phi_1(\hat{T}_i)$ at most once, say at p_i . For each p_i , we may assume $p_i \in \Gamma_{\Phi_1(\hat{T}_i)}$ and the isotopic copies \hat{S}'_j of \hat{S}_j , $j = 1, 2$, which contain $\hat{\gamma}'$ are convex with *efficient Legendrian boundary* (i.e., efficient with respect to the torus containing the boundary component). In order to compare the twisting number of $\hat{\gamma}'$ cut open along $\cup_i \Phi_1(\hat{T}_i)$ and that of $\hat{\gamma}$ cut open along $\cup_i \hat{T}_i$, we need to modify the cut-open $\hat{\gamma}$ slightly by attaching unfolding layers onto $W \setminus (\cup_i \hat{T}_i)$ and extending this cut-open arc slightly on the attached toric annulus without modifying the twisting number. Once we do this, we are comparing (the modified) $\hat{\gamma}$ with $\hat{\gamma}'$, both on $W \setminus (\cup_i \Phi_1(\hat{T}_i))$, which reduces the problem to the torus boundary case. \square

Completion of proof of Theorem 0.2. Theorem 4.8 and its proof imply Theorem 0.2, once we look at the mapping class group $\pi_0(\text{Diff}(M))$ of the closed toroidal manifold M . Our argument is a little different from that of Colin [4] in that we do not bound the *torsion*, and instead use facts about the mapping class group to pass from “infinitely many isotopy classes” to “infinitely many isomorphism classes”. Let T_1, \dots, T_m be the incompressible tori as in the Torus Decomposition Theorem and M_1, \dots, M_n the connected components of $M \setminus (\cup_{i=1}^m T_i)$. Apply Gabai’s sutured manifold decomposition to each M_j , obtain the corresponding universally tight contact structure on M_j , and add appropriate $(T^2 \times I, \xi_{k_i})$ along each T_i . This is $\zeta = \zeta(k_1, \dots, k_m)$.

Note that any element of $\text{Diff}(M)$ must fix the isotopy class of $\cup_{i=1}^m T_i$, but may permute the T_1, \dots, T_m (as well as M_1, \dots, M_n). Since there are only finitely many such permutations, we may assume that the elements of $\text{Diff}(M)$ we consider fix each connected component M_j (as well as T_i). The hyperbolic components M_j present no problem, since $\pi_0(\text{Diff}(M_j))$ is finite modulo Dehn twists on the boundary, according to Theorem 4.5. This means that, for any compact surface S with boundary on ∂M_j , there are finitely many isotopy classes of images of S under $\text{Diff}(M_j)$, up to isotoping ∂S along ∂M_j . Let M_j and M_{j+1} be adjacent hyperbolic components with common boundary T_i , and S_j, S_{j+1} be norm-minimizing surfaces in M_j, M_{j+1} with boundary on T_i . Both S_j and S_{j+1} are not disks or annuli (by Lemma 4.4). Then take curves γ_j on S_j and γ_{j+1} on S_{j+1} to form a closed curve γ as in the proof of Theorem 4.8. Since S_j, S_{j+1} are not disks or annuli, we may take γ_j, γ_{j+1} to be nontrivial curves which begin and end on the same boundary component of S_j, S_{j+1} , respectively. Now consider $t(\mathcal{L}_\gamma)$ as well as $t(\mathcal{L}_{\phi(\gamma)})$, for all $\phi \in \pi_0(\text{Diff}(M))$. Since there are finitely many isotopy classes $\phi(\gamma)$ modulo Dehn twists along tori, and Dehn twisting does not change $t(\mathcal{L}_\gamma)$, there exists a finite number $-k_\gamma = \inf_{[\phi] \in \pi_0(\text{Diff}(M))} t(\mathcal{L}_{\phi(\gamma)})$. In a manner similar to Proposition 4.9, we can prove that $t(\mathcal{L}_\gamma) = -(k_i + 1)$. (Note that our current situation is not quite the same as the situation in Proposition 4.9, but the proof goes through, if we first extricate an isotopic copy of γ from components M_s with $s \neq j, j+1$. This extrication can be done as before without penalty, by passing to a large finite cover.) If we inductively choose the next $\zeta = \zeta(k_1, \dots, k_m)$ so that each $k_i \gg k_\gamma$, the new ζ cannot be isomorphic to the ones previously chosen.

On the other hand, the Seifert fibered components M_j have larger mapping class groups and require a little more care. Let $\pi_j : M_j \rightarrow B_j$ be the Seifert fibration map which projects to the base B_j . In case M_j is toroidal — see the paragraph after Theorem 4.2 for exceptions — the torus T is vertical, and there exists another torus T' which intersects T *persistently*. This case is already treated in Colin [4], where the torsion for T and T' are shown to be finite. The two remaining cases are: B_j is S^2 with at most three singular points or punctures, or B_j is \mathbb{RP}^2 with at most two singular points or punctures.

In the S^2 case, if there are two punctures and no singular points, then M_j is $T^2 \times I$ and the minimality of the Torus Decomposition of M implies that M must be a torus bundle over S^1 , in which case the theorem is already proved by Giroux [11]. If there is one puncture and one or no singular points, then the torus is compressible. Therefore, for S^2 , we assume that there is a total of three singular points or punctures (with at least one puncture). For \mathbb{RP}^2 , if there is one puncture and no singular points, we have a twisted I -bundle over a Klein bottle which is separated by the torus boundary. We will treat this case separately.

Let S_j be a norm-minimizing, oriented, surface on M_j that is not homologous to a vertical annulus. It follows that S_j may be made horizontal, and by the Riemann-Hurwitz formula, S_j cannot be an annulus or a disk. A diffeomorphism ϕ of M_j must take S_j to another horizontal surface $\phi(S_j)$, since S_j is not an annulus and hence cannot be vertical (Proposition 4.3). A Riemann-Hurwitz formula calculation reveals that $\pi : \phi(S_j) \rightarrow B_j$ and $\pi : S_j \rightarrow B_j$ have the same number of sheets. Moreover, ϕ is isotopic to a diffeomorphism which preserves the Seifert fibration. Now, if we construct γ as before from adjacent M_j and M_{j+1} (M_{j+1} not necessarily Seifert fibered), then $t(\mathcal{L}_\gamma)$ remains invariant under various choices of S_j up to diffeomorphism. This is because a diffeomorphism ϕ of M_j which does not permute

the boundary components will induce a map $T_i = \mathbb{R}^2/\mathbb{Z}^2 \rightarrow T_i = \mathbb{R}^2/\mathbb{Z}^2$ which sends $(x, y) \mapsto (x, px + y)$, $p \in \mathbb{Z}$, on each boundary component T_i . The rest of the argument goes through as in the hyperbolic case.

Finally assume M_j is a twisted I -bundle over a Klein bottle and $T_i = \partial M_j$. In this case we may distinguish the universally tight contact structures up to isomorphism by passing to a double cover \widetilde{M} of M which is obtained by gluing two copies of $M \setminus M_j$ onto the double cover $T^2 \times I$ of M_j . (We may need to take larger covers to unwind all the twisted I -bundle components over Klein bottles.)

The contact structures on M we construct lift to nonisomorphic universally tight contact structures on \widetilde{M} , and therefore it follows that there are infinitely many universally tight contact structures on M up to isomorphism. \square

4.4. Construction of auxiliary manifold W . In this section we construct W satisfying the properties stipulated in the previous section.

Case 1. Suppose that T is nonseparating and the next stage of the Gabai decomposition comes from a connected surface S which intersects both T_+ and T_- . Let $\gamma \subset S$ be an arc from T_+ to T_- . First use residual finiteness to take a large finite Galois cover $\pi : \widetilde{M} \rightarrow M$ which expands T . Let $\widetilde{T}_1, \widetilde{T}_2$ be connected components of $\pi^{-1}(T)$, and $\widetilde{\gamma} \subset \widetilde{M}$ a lift of γ with one endpoint on \widetilde{T}_1 and the other endpoint on \widetilde{T}_2 . (By a “lift” $\widetilde{\gamma}$ of γ we mean an arc in \widetilde{M} which maps 1-1 down to γ via π , with the exception of endpoints of the arc.) It might happen that $\widetilde{T}_1 = \widetilde{T}_2$. In that case, we use residual finiteness again to unwrap in the direction transverse to $\widetilde{T}_1 = \widetilde{T}_2$, so that the two connected components are distinct. Now, cut \widetilde{M} along \widetilde{T}_1 and \widetilde{T}_2 and glue together pairs $\widetilde{T}_{1,+}, \widetilde{T}_{2,-}$ and $\widetilde{T}_{1,-}, \widetilde{T}_{2,+}$ in a $\pi_1(T)$ -equivariant manner so that the endpoints of $\widetilde{\gamma}$ glue to give a closed curve $\widehat{\gamma}$ in the manifold W . We may need to throw away a component which does not contain $\widehat{\gamma}$. Note that, in order to glue the boundaries, we need \widetilde{M} to be a *Galois* cover of M — i.e., $\pi_1(M)$ must act transitively on \widetilde{M} . The glued-up contact structure ζ is universally tight by Theorem 4.7.

Case 2. Either T is separating and there exist two surfaces S_1 and S_2 or T is nonseparating and there are two surfaces S_1 and S_2 which bound T_+, T_- respectively. Let γ_1, γ_2 be arcs on S_1 and S_2 which glue to give γ . Assume γ_i are closed curves (identify their endpoints on T). Let M_i be the component of $M \setminus T$ containing S_i (note M_1 may be the same as M_2).

Lemma 4.10. $\gamma_i \notin \pi_1(T)$, $i = 1, 2$, as elements of $\pi_1(M)$.

Proof. Consider the universal cover $\overline{\pi} : \mathbb{R}^3 = \overline{M}_i \rightarrow M_i$. We claim that a lift $\overline{\gamma}$ of $\gamma = \gamma_i$ has endpoints on different \mathbb{R}^2 -components of $\overline{\pi}^{-1}(T)$. (Call them \mathbb{R}_j^2 , $j = 1, 2$.) Assume otherwise; namely the universal cover \overline{S} (connected component) of $S = S_i$ intersects a particular \mathbb{R}_j^2 along at least two coherently oriented lines. This follows from the well-grooming of S — all the components of $\partial S \cap T$ were oriented in the same direction. Now, \overline{S} must separate $\overline{M}_i = \mathbb{R}^3$ into two half-planes. However, the coherent orientation of $\overline{S} \cap \mathbb{R}_i^2$ contradicts the fact that \overline{S} separates. This implies that $\gamma \notin \pi_1(T) \subset \pi_1(M_i)$. To conclude the proof, we simply note that $\pi_1(M_i)$ injects into $\pi_1(M)$. \square

Now we use the fact that $\pi_1(T)$ is separable in $\pi_1(M)$ (Theorem 4.1). There exist finite index subgroups K_1, K_2 in $\pi_1(M)$ which contain $\pi_1(T)$ but miss γ_1, γ_2 , respectively. The finite cover corresponding to $K_1 \cap K_2$ will then satisfy the condition that the endpoints of lifts of γ_1 lie on distinct connected torus components of the preimage of T , and the same holds for γ_2 . Finally take a finite index normal subgroup $K \subset K_1 \cap K_2$ which sufficiently expands T . Let $\pi : \widetilde{M} \rightarrow M$ be the corresponding finite cover and $\widetilde{\gamma} = \widetilde{\gamma}_1 + \widetilde{\gamma}_2$ a lift of γ . Here, let $\widetilde{T}_j, j = 1, 2, 3$, be three components of $\pi^{-1}(T)$, and $\widetilde{\gamma}_i, i = 1, 2$, connect from \widetilde{T}_i to \widetilde{T}_{i+1} . (The endpoint of $\widetilde{\gamma}_1$ equals the initial point of $\widetilde{\gamma}_2$.) Now cut \widetilde{M} along \widetilde{T}_1 and \widetilde{T}_3 and glue pairs $\widetilde{T}_{1,+}, \widetilde{T}_{3,-}$ and $\widetilde{T}_{1,-}, \widetilde{T}_{3,+}$ equivariantly with respect to $\pi_1(T)$ so that $\widetilde{\gamma}$ glues into a closed curve $\widehat{\gamma}$. This is the desired manifold W .

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